HYPERSPHERICAL REALIZATION OF A UNIT NORM CONDITION

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Introduction. A device standard to orthodox quantum mechanics is the *density matrix*

$$\boldsymbol{\rho} = p_1|\psi_1)(\psi_1| + p_2|\psi_2)(\psi_2| + \cdots + p_n|\psi_n)(\psi_n|$$

which serves to describe a statistical mixture of quantum states. A point seldom remarked is that *distinct mixtures can give rise to the same density matrix*, and have therefore to be considered physically equivalent. Tom Wieting has provided an elegant geometrical characterization of all mixtures equivalent to the 2-state mixture

$$\boldsymbol{\rho} = p_1|\psi_1)(\psi_1| + p_2|\psi_2)(\psi_2|$$

The following material was intended to support my effort to describe the n-state generalization of Wieting's construction. But it involves notions that are so primitive, and which see service in such an extraordinary variety of applications, that it may be of interest to readers who do not share my interest in the relatively esoteric physical problem from which it sprang (and which made natural my mode of approach). My consistent effort will be to keep simple things simple.

Let ξ be a complex n-vector:

$$\xi = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$$

Such an object has $2n$ degrees of freedom. Impose the "unit norm condition"

$$\xi^{\mathsf{t}}\xi \equiv \bar{u}_1 u_1 + \bar{u}_2 u_2 + \cdots + \bar{u}_n u_n = 1$$

The resulting "unit vectors" $\hat{\xi}$ are members of a $(2n-1)$-parameter population. Each $\hat{\xi}$ drawn from that population supports a "ray" consisting of all complex multiples of $\hat{\xi}$:

$$\xi \in \hat{\xi}\text{-ray if and only if } \xi = re^{i\alpha}\cdot\hat{\xi}$$

Evidently there is a $(2n-2)$-parameter population of such rays, each of which is by nature a $\mathcal{C}_1 \subset \mathcal{C}_n$. We observe finally that

$$\hat{\xi} \text{ and } \hat{\xi}' \text{ support the same ray if and only if } \textit{phase equivalent}\text{: } \hat{\xi}' = e^{i\alpha}\cdot\hat{\xi}$$

Look now to properties of the matrix

$$\mathbb{P} \equiv \begin{pmatrix} u_1\bar{u}_1 & u_1\bar{u}_2 & \dots & u_1\bar{u}_n \\ u_2\bar{u}_1 & u_2\bar{u}_2 & \dots & u_2\bar{u}_n \\ \vdots & \vdots & & \vdots \\ u_n\bar{u}_1 & u_n\bar{u}_2 & \dots & u_n\bar{u}_n \end{pmatrix}$$

which has been assembled from the elements of the unit vector $\hat{\xi}$. The matrix is manifestly hermitian

$$\mathbb{P}^{\mathsf{t}} = \mathbb{P}$$

and is readily seen to be projective

$$\mathbb{P}^2 = \mathbb{P}$$

From

$$\mathrm{tr}\,\mathbb{P} = 1$$

we learn that $\mathbb{P}$ projects onto a 1-dimensional space, while from

$$\mathbb{P}\hat{\xi} = \hat{\xi}$$

we see that it projects in fact onto the $\hat{\xi}$-ray. In the latter connection we notice that each of the phase-equivalent unit vectors $\hat{\xi}' = e^{i\alpha} \cdot \hat{\xi}$ gives rise to the same projector $\mathbb{P}$.

The matrix

$$\mathbb{P}_{\perp} \equiv \mathbb{I} - \mathbb{P}$$

is also projective. From

$$\mathrm{tr}\,\mathbb{P}_{\perp} = n - 1$$

we learn that $\mathbb{P}_{\perp}$ projects onto a $(n-1)$-dimensional space, which by

$$\mathbb{P}_{\perp}\mathbb{P} = \mathbb{O}$$

is the $\mathcal{C}_{n-1}$ orthogonal to the $\hat{\xi}$-ray.

It is mainly to describe my objective and general plan of attack, and only incidentally to recover some famous specific results, that I look now to the simplest non-trivial case, which is...

1. The 2-dimensional case. To "mechanize" the norm condition $\bar{u}_1 u_1 + \bar{u}_2 u_2 = 1$ write

$$\begin{aligned} \bar{u}_1 u_1 &= \cos^2\phi \\ \bar{u}_2 u_2 &= \sin^2\phi \end{aligned}$$

Then

$$\begin{aligned} u_1 &= e^{i\alpha}\cos\phi \\ u_2 &= e^{i\beta}\sin\phi \end{aligned}$$

One can without loss of generality assume that $\alpha + \beta = 0$; to phrase the issue another way, one can always

$$\text{write} \quad \begin{matrix} \alpha = \text{phase} + \psi \\ \beta = \text{phase} - \psi \end{matrix} \quad \text{and look to the case: phase} = 0$$

Assuming this to have been done, we have

$$\hat{\xi} = \begin{pmatrix} e^{+i\psi}\cos\phi \\ e^{-i\psi}\sin\phi \end{pmatrix}$$

The associated projector reads

$$\begin{aligned} \mathbb{P} &= \begin{pmatrix} \cos^2\phi & e^{+2i\psi}\cos\phi\sin\phi \\ e^{-2i\psi}\cos\phi\sin\phi & \sin^2\phi \end{pmatrix} \\ &= \tfrac{1}{2}\begin{pmatrix} 1+\cos 2\phi & e^{+2i\psi}\sin 2\phi \\ e^{-2i\psi}\sin 2\phi & 1-\cos 2\phi \end{pmatrix} \end{aligned}$$

A cultivated sense of tidy good-housekeeping—a modest virtue far short of genius—inspires the observation that the result just achieved can be notated

$$\begin{aligned} \mathbb{P} &= \tfrac{1}{2}\begin{pmatrix} 1 \;+\; \hat{a}_1 & \hat{a}_2 + i\hat{a}_3 \\ \hat{a}_2 - i\hat{a}_3 & 1 \;-\; \hat{a}_1 \end{pmatrix} \\ &= \tfrac{1}{2}\big\{\mathbb{I} + \hat{a}_1\mathbb{S}_1 + \hat{a}_2\mathbb{S}_2 + \hat{a}_3\mathbb{S}_3\big\} \\ &= \mathbb{P}_{\hat{\boldsymbol{a}}} \quad : \quad \text{more explicit notation} \end{aligned}$$

with

$$\mathbb{S}_1 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathbb{S}_2 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathbb{S}_3 \equiv \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

and

$$\begin{aligned} \hat{a}_1 &\equiv \cos 2\phi \\ \hat{a}_2 &\equiv \sin 2\phi\cos 2\psi \\ \hat{a}_3 &\equiv \sin 2\phi\,\sin 2\psi \end{aligned}$$

The traceless 2×2 hermitian matrices $\big\{\mathbb{S}_1, \mathbb{S}_2, \mathbb{S}_3\big\}$ are none other than the familiar "Pauli matrices" (notated $\mathbb{S}$ because I do not have typographic access to a "blackboard sigma"), while the real numbers $\big\{\hat{a}_1, \hat{a}_2, \hat{a}_3\big\}$ satisfy

$$\hat{a}_1^2 + \hat{a}_2^2 + \hat{a}_3^2 = 1$$

and—thought of as components of a real 3-vector

$$\hat{\boldsymbol{a}} \equiv \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \\ \hat{a}_3 \end{pmatrix}$$

—can be considered to mark a point on the surface of the unit sphere in 3-dimensional $\boldsymbol{a}$-space, a point of which $\big\{2\phi, 2\psi\big\}$ are (relative to a somewhat eccentric convention: see the figure) the spherical coordinates.

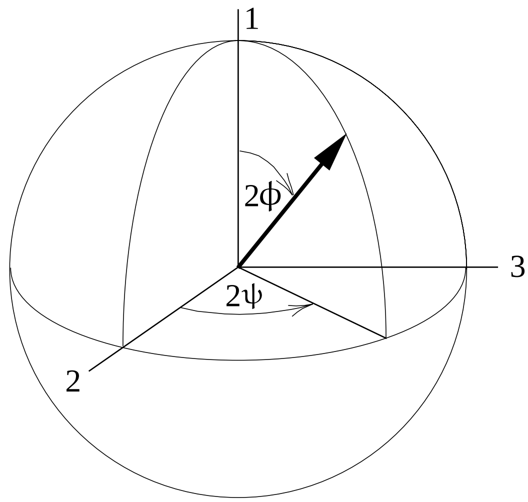

FIGURE 1: *Non-standard spherical coordinate system employed in the text.*

In notation now available to us we have

$$\begin{aligned}\mathbb{P}_\perp &= \mathbb{I} - \tfrac{1}{2}\big\{\mathbb{I} + \hat a_1\,\mathbb{S}_1 + \hat a_2\,\mathbb{S}_2 + \hat a_3\,\mathbb{S}_3\big\}\\ &= \tfrac{1}{2}\big\{\mathbb{I} - \hat a_1\,\mathbb{S}_1 - \hat a_2\,\mathbb{S}_2 - \hat a_3\,\mathbb{S}_3\big\}\\ &= \mathbb{P}_{-\hat{\boldsymbol{a}}}\end{aligned}$$

which projects onto the ray *orthogonal* to the $\hat\xi$-ray. We have achieved (by no act of genius: it has simply fallen into our lap) an association

$$\text{point } \hat{\boldsymbol{a}} \text{ on the unit 3-sphere } \longleftrightarrow \hat\xi\text{-ray}$$

with the property that if ξ associates with $\hat{\boldsymbol{a}}$ then $\xi_\perp$ associates with the antipodal point $-\hat{\boldsymbol{a}}$.

If the 2×2 matrix $\mathbb{U}$ is unitary

$$\mathbb{U}^{\mathsf{t}}\mathbb{U} = \mathbb{I} \quad : \quad \text{symbolized } \mathbb{U}\in U(2)$$

then $\hat\xi \longrightarrow \hat\xi' \equiv \mathbb{U}\hat\xi$ is norm-preserving, and the induced linear transformation $\hat{\boldsymbol{a}} \longrightarrow \hat{\boldsymbol{a}}' \equiv \mathbb{R}\,\hat{\boldsymbol{a}}$ is necessarily rotational:

$$\mathbb{R}^{\mathsf{T}}\mathbb{R} = \mathbb{I} \quad : \quad \text{symbolized } \mathbb{R}\in O(3)$$

Looking now to some of the details implicit in that remark: every unitary $\mathbb{U}$ can be displayed

$$\mathbb{U} = e^{i(\text{hermitian})}$$

which entails

$$\det \mathbb{U} = e^{\text{tr}\{i(\text{hermitian})\}}$$

so $\mathbb{U}$ will be *unimodular* (an element of the subgroup $SU(2) \subset U(2)$) if and only if the exponentiated hermitian matrix is traceless. In all such cases we can write

$$\mathbb{U} = e^{i\theta\{\lambda_1\mathbb{S}_1+\lambda_2\mathbb{S}_2+\lambda_3\mathbb{S}_3\}} \quad : \quad \lambda_1^2+\lambda_2^2+\lambda_3^2 = 1$$

where

$$\lambda_1\mathbb{S}_1+\lambda_2\mathbb{S}_2+\lambda_3\mathbb{S}_3 \equiv \mathbb{H} = \begin{pmatrix} \lambda_1 & \lambda_2+i\lambda_3 \\ \lambda_2-i\lambda_3 & -\lambda_1 \end{pmatrix}$$

is traceless hermitian and has evidently the property that $\det(\mathbb{H}-\lambda\mathbb{I}) = \lambda^2-1$. By the Cayley-Hamilton theorem we therefore have $(i\mathbb{H})^2+\mathbb{I} = \mathbb{O}$, giving

$$\mathbb{U} = \cos\theta\cdot\mathbb{I} + i\sin\theta\cdot\big\{\lambda_1\mathbb{S}_1+\lambda_2\mathbb{S}_2+\lambda_3\mathbb{S}_3\big\}$$

To obtain the structure of the associated rotation matrix $\mathbb{R}$ we note that comparison of these two representations of $\mathbb{P}_{\hat{\boldsymbol{a}}}$

$$\begin{pmatrix} u_1\bar u_1 & u_1\bar u_2 \\ u_2\bar u_1 & u_2\bar u_2 \end{pmatrix} \quad \longleftarrow \text{compare} \longrightarrow \quad \tfrac{1}{2}\begin{pmatrix} 1\ +\ \hat a_1 & \hat a_2+i\hat a_3 \\ \hat a_2-i\hat a_3 & 1\ -\ \hat a_1 \end{pmatrix}$$

gives

$$\begin{aligned} 1 &= \quad u_1\bar u_1+u_2\bar u_2 &&= \hat\xi^{\mathsf{t}}\,\mathbb{S}_0\,\hat\xi \quad \text{with } \mathbb{S}_0\equiv\mathbb{I} \\ \hat a_1 &= \quad u_1\bar u_1-u_2\bar u_2 &&= \hat\xi^{\mathsf{t}}\,\mathbb{S}_1\,\hat\xi \\ \hat a_2 &= \quad u_1\bar u_2+u_2\bar u_1 &&= \hat\xi^{\mathsf{t}}\,\mathbb{S}_2\,\hat\xi \\ \hat a_3 &= -i(u_1\bar u_2-u_2\bar u_1) &&= \hat\xi^{\mathsf{t}}\,\mathbb{S}_3\,\hat\xi \end{aligned}$$

Evidently

$$\begin{aligned} \hat a_1' &= \hat\xi^{\,\mathsf{t}}\mathbb{U}^{\mathsf{t}}\mathbb{S}_1\mathbb{U}\hat\xi \\ \hat a_2' &= \hat\xi^{\,\mathsf{t}}\mathbb{U}^{\mathsf{t}}\mathbb{S}_2\mathbb{U}\hat\xi \\ \hat a_3' &= \hat\xi^{\,\mathsf{t}}\mathbb{U}^{\mathsf{t}}\mathbb{S}_3\mathbb{U}\hat\xi \end{aligned}$$

But by appropriation of a general identity we have

$$\mathbb{U}^{\mathsf{t}}\,\mathbb{S}\,\mathbb{U} = e^{-i\theta\mathbb{H}}\,\mathbb{S}\,e^{+i\theta\mathbb{H}} = \sum_{k=0}^{\infty}\tfrac{1}{k!}\big\{(-i\theta\mathbb{H})^k,\mathbb{S}\big\}$$

where $\big\{\mathbb{A}^k,\mathbb{B}\big\} \equiv [\mathbb{A},[\mathbb{A},\ldots,[\mathbb{A},\mathbb{B}]\ldots]]$ is a k-fold "nested commutator." From the familiarly "quaternionic" multiplicative properties of the Pauli matrices

$$\mathbb{S}_j\mathbb{S}_k = \delta_{jk}\,\mathbb{I} - i\epsilon_{jkl}\,\mathbb{S}_l$$

it follows readily that $[\mathbb{S}_j,\mathbb{S}_k] = -2i\epsilon_{jkl}\mathbb{S}_l$, and therefore that

$$\Big[(-i\theta\mathbb{H}), \begin{pmatrix}\mathbb{S}_1\\ \mathbb{S}_2\\ \mathbb{S}_3\end{pmatrix}\Big] = 2\theta\begin{pmatrix} 0 & -\lambda_3 & \lambda_2 \\ \lambda_3 & 0 & -\lambda_1 \\ -\lambda_2 & \lambda_1 & 0 \end{pmatrix}\begin{pmatrix}\mathbb{S}_1\\ \mathbb{S}_2\\ \mathbb{S}_3\end{pmatrix}$$

Assembly of facts now in hand gives

$$\hat{\boldsymbol{a}} \longrightarrow \hat{\boldsymbol{a}}' = \mathbb{R}\,\hat{\boldsymbol{a}}$$

where

$$\mathbb{R} = e^{2\theta\mathbb{A}} \quad : \quad \mathbb{A} \equiv \begin{pmatrix} 0 & -\lambda_3 & \lambda_2 \\ \lambda_3 & 0 & -\lambda_1 \\ -\lambda_2 & \lambda_1 & 0 \end{pmatrix} \text{ real antisymmetric}$$

describes a rotation through angle 2θ about the axis defined by the unit 3-vector $\boldsymbol{\lambda}$. The exponentiated 2-factor signals the double-valuedness of the spinor representations of $O(3)$: the angular advancement

$$2\theta \to 2\theta + 2\pi \quad \text{sends} \quad \mathbb{R} \to \mathbb{R} \quad \text{but} \quad \mathbb{U} \to -\mathbb{U}$$

so one has to execute a second complete rotation in 3-space to return $\mathbb{U}$ to its original value.

All of which is familiar water, much trafficked. But by launching our canoe from a seldom visited beach we have managed simply to drift with the current, from landmark to famous landmark, while scarcely lifting a paddle. The question is: Does the procedure admit of dimensional generalization?

I will discuss the utility, in climbing to higher dimension, of an "extension ladder" technique, the essence of which becomes evident when (in—for example —the 5-dimensional case) one writes

$$\xi^{\mathsf{t}}\xi = (((\bar{u}_1 u_1 + \bar{u}_2 u_2) + \bar{u}_3 u_3) + \bar{u}_4 u_4) + \bar{u}_5 u_5$$

The procedure will result in what can, from several points of view, be thought of as "hierarchically nested copies of the 2-dimensional theory," but is, it should be recognized from the outset, flawed by a high degree of structural asymmetry: the slots provided by the expression

$$\xi^{\mathsf{t}}\xi = (\cdots(((\bullet + \bullet) + \bullet) + \bullet) + \cdots + \bullet) + \bullet$$

can be filled in $n!$ distinct ways, and give rise to $(n-1)!$ distinct variants of the same theory.

2. Climbing to higher dimension. Looking initially to the 3-dimensional case: to mechanize the norm condition

$$(\bar{u}_1 u_1 + \bar{u}_2 u_2) + \bar{u}_3 u_3 = 1$$

we (STEP ONE) write

$$\begin{aligned} (\bar{u}_1 u_1 + \bar{u}_2 u_2) &= \cos^2\phi_2 \\ \bar{u}_3 u_3 &= \sin^2\phi_2 \end{aligned}$$

and then (STEP TWO) write

$$\begin{aligned} \bar{u}_1 u_1 &= \cos^2\phi_2 \cos^2\phi_1 \\ \bar{u}_2 u_2 &= \cos^2\phi_2 \sin^2\phi_1 \end{aligned}$$

At this point we have, in effect, simply introduced spherical coordinates to identify points on the unit sphere in 3-dimensional "modulus space." It now follows that

$$\begin{aligned} u_1 &= e^{i\alpha}\cos\phi_2\cos\phi_1\\ u_2 &= e^{i\beta}\cos\phi_2\sin\phi_1\\ u_3 &= e^{i\gamma}\sin\phi_2 \end{aligned}$$

which give back their 2-dimensional counterparts at $\phi_2 = 0$. Next we

$$\text{write}\quad \begin{aligned} \alpha &= \text{phase} + \psi_2 + \psi_1\\ \beta &= \text{phase} + \psi_2 - \psi_1\\ \gamma &= \text{phase} - \psi_2 \end{aligned} \quad \text{and look to the case: phase} = 0$$

This done , we obtain

$$\hat{\xi} = \begin{pmatrix} e^{i(+\psi_2+\psi_1)}\cos\phi_2\cos\phi_1\\ e^{i(+\psi_2-\psi_1)}\cos\phi_2\sin\phi_1\\ e^{i(-\psi_2\qquad)}\sin\phi_2 \end{pmatrix}$$

and notice that the numerology is correct: we have now in hand a 4-parameter characterization of the rays in $\mathcal{C}_3$.[1] In service of clarity, write

$$\hat{\xi} = \begin{pmatrix} e^{+i\psi_2}\cos\phi_2\begin{pmatrix} e^{+i\psi_1}\cos\phi_1\\ e^{-i\psi_1}\sin\phi_1\end{pmatrix}\\ e^{-i\psi_2}\sin\phi_2 \end{pmatrix}$$

which in the 4-dimensional case becomes

$$\hat{\xi} = \begin{pmatrix} e^{+i\psi_3}\cos\phi_3\begin{pmatrix} e^{+i\psi_2}\cos\phi_2\begin{pmatrix} e^{+i\psi_1}\cos\phi_1\\ e^{-i\psi_1}\sin\phi_1\end{pmatrix}\\ e^{-i\psi_2}\sin\phi_2 \end{pmatrix}\\ e^{-i\psi_3}\sin\phi_3 \end{pmatrix}$$

and exposes clearly the essence of the "extension ladder" technique.

One can—by "lowering the ladder"—easily read off the angular parameters characteristic of any given $\hat{\xi}$; the technique, as experienced in the 4-dimensional case, is illustrated below:

$$\begin{aligned} &\text{from}\quad u_4 = e^{-i\psi_3}\sin\phi_3 \quad \text{read off } \phi_3 \text{ and } \psi_3\\ &\text{from}\quad \frac{u_3}{e^{+i\psi_3}\cos\phi_3} = e^{-i\psi_2}\sin\phi_2 \quad \text{read off } \phi_2 \text{ and } \psi_2\\ &\text{from}\quad \frac{u_2}{e^{+i\psi_3}\cos\phi_3\cdot e^{+i\psi_2}\cos\phi_2} = e^{-i\psi_1}\sin\phi_1 \quad \text{read off } \phi_1 \text{ and } \psi_1 \end{aligned}$$

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[1] In the 4-dimensional case the analogous equation would read

$$\hat{\xi} = \begin{pmatrix} u_1\\ u_2\\ u_3\\ u_4 \end{pmatrix} = \begin{pmatrix} e^{i(+\psi_3+\psi_2+\psi_1)}\cos\phi_3\cos\phi_2\cos\phi_1\\ e^{i(+\psi_3+\psi_2-\psi_1)}\cos\phi_3\cos\phi_2\sin\phi_1\\ e^{i(+\psi_3-\psi_2\qquad)}\cos\phi_3\sin\phi_2\\ e^{i(-\psi_3\qquad\qquad)}\sin\phi_3 \end{pmatrix}$$

It is to facilitate use of the "extension ladder" that we have contrived to have the design of the exponents mimic the design of the factors.

In a simplified notation we have

$$\hat{\xi} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} q_1 p_1 \\ q_1 p_2 \\ q_2 \end{pmatrix} \quad : \quad \text{3-dimensional case}$$

$$\hat{\xi} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} r_1 q_1 p_1 \\ r_1 q_1 p_2 \\ r_1 q_2 \\ r_2 \end{pmatrix} \quad : \quad \text{4-dimensional case}$$

$$\vdots$$
$$\text{etc.}$$

and associated projectors of the design

$$\mathbb{P} = \begin{pmatrix} q_1 p_1 \bar{q}_1 \bar{p}_1 & q_1 p_1 \bar{q}_1 \bar{p}_2 & q_1 p_1 \bar{q}_2 \\ q_1 p_2 \bar{q}_1 \bar{p}_1 & q_1 p_2 \bar{q}_1 \bar{p}_2 & q_1 p_2 \bar{q}_2 \\ q_2 \bar{q}_1 \bar{p}_1 & q_2 \bar{q}_1 \bar{p}_2 & q_2 \bar{q}_2 \end{pmatrix} \quad \text{with} \quad \begin{matrix} p_1 \bar{p}_1 + p_2 \bar{p}_2 = 1 \\ q_1 \bar{q}_1 + q_2 \bar{q}_2 = 1 \end{matrix}$$

$$\vdots$$
$$\text{etc.}$$

While $\mathbb{P}$ projects onto the $\hat{\xi}$-ray, $\mathbb{P}_\perp \equiv \mathbb{I} - \mathbb{P}$ projects (as previously noted) onto the $\mathcal{C}_{n-1}$ orthogonal to $\hat{\xi}$. There exists, however, a fairly natural way to introduce an orthonormal basis $\{\hat{\xi}_1, \hat{\xi}_2, \ldots \hat{\xi}_{n-1}\}$ into $\mathcal{C}_{n-1}$, a fairly natural way to "resolve" $\mathbb{P}_\perp$:

$$\mathbb{P}_\perp = \mathbb{P}_1 + \mathbb{P}_2 + \cdots + \mathbb{P}_{n-1}$$

The construction I have in mind (described below) makes nested use of results special to the 2-dimensional case, which I digress now to assemble and recast.

We learned in §1 to associate

$$\begin{array}{r} \text{unit 3-vector } \hat{\boldsymbol{a}} \\ \updownarrow \\ 2\times 2 \text{ projector } \mathbb{P}_{\hat{\boldsymbol{a}}} \\ \updownarrow \\ \hat{\xi}\text{-ray onto which } \mathbb{P}_{\hat{\boldsymbol{a}}} \text{ projects} \end{array}$$

More specifically, we have

$$\begin{aligned} \hat{\boldsymbol{a}} \longleftrightarrow \hat{\xi} \;\; &= \begin{pmatrix} e^{+i\psi} \cos\phi \\ e^{-i\psi} \sin\phi \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \\ -\hat{\boldsymbol{a}} \longleftrightarrow \hat{\xi}_\perp &= \begin{pmatrix} e^{+i(\psi+\frac{\pi}{2})} \cos(\frac{\pi}{2}-\phi) \\ e^{-i(\psi+\frac{\pi}{2})} \sin(\frac{\pi}{2}-\phi) \end{pmatrix} \\ &= i \begin{pmatrix} +e^{+i\psi} \sin\phi \\ -e^{+i\psi} \cos\phi \end{pmatrix} = i \begin{pmatrix} +\bar{u}_2 \\ -\bar{u}_1 \end{pmatrix} \end{aligned}$$

Adopting language specific to the 4-dimensional case to make my point: it becomes natural in this light to proceed

$$\hat{\xi} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} r_1q_1p_1 \\ r_1q_1p_2 \\ r_1q_2 \\ r_2 \end{pmatrix} \quad : \quad \begin{cases} \text{associate } \hat{\boldsymbol{a}}_1 \leftrightarrow \ \text{ innermost 2-vector } \binom{p_1}{p_2} \\ \text{associate } \hat{\boldsymbol{a}}_2 \leftrightarrow \ \text{ next-inner 2-vector } \binom{q_1}{q_2} \\ \text{associate } \hat{\boldsymbol{a}}_3 \leftrightarrow \ \text{ outermost 2-vector } \binom{r_1}{r_2} \end{cases}$$

and to adopt notation which in the n-dimensional case reads

$$\hat{\xi} = \big\{\hat{\boldsymbol{a}}_{n-1}, \ldots, \hat{\boldsymbol{a}}_2, \hat{\boldsymbol{a}}_1\big\}$$

We have been led thus to associate rays in $\mathcal{C}_n$ not (as might plausibly have been anticipated) with hyperspheres of some description, but with "3-spheres decorated with. . . 3-spheres decorated with 3-spheres."

In 2-dimensional theory we have

$$\big\{\hat{\boldsymbol{a}}_1\big\} \perp \big\{-\hat{\boldsymbol{a}}_1\big\}$$

Looking to the 3-dimensional theory, it becomes in this light fairly natural to notice that

$$\big\{\hat{\boldsymbol{a}}_2, \hat{\boldsymbol{a}}_1\big\} \perp \big\{-\hat{\boldsymbol{a}}_2, \hat{\boldsymbol{a}}_1\big\} \quad \text{i.e.,} \quad \begin{pmatrix} q_1p_1 \\ q_1p_2 \\ q_2 \end{pmatrix} \perp i \begin{pmatrix} +\bar{q}_2p_1 \\ +\bar{q}_2p_2 \\ -\bar{q}_1 \end{pmatrix}$$

which is, in fact, clear by inspection. To obtain a third vector simultaneously orthogonal to $\big\{\hat{\boldsymbol{a}}_2, \hat{\boldsymbol{a}}_1\big\}$ and $\big\{-\hat{\boldsymbol{a}}_2, \hat{\boldsymbol{a}}_1\big\}$ I borrow a trick (and notation) from ordinary vector algebra: writing

$$\hat{\boldsymbol{\xi}} \equiv \big\{\hat{\boldsymbol{a}}_2, \hat{\boldsymbol{a}}_1\big\} \quad \text{and} \quad \hat{\boldsymbol{\xi}}_\perp \equiv \big\{-\hat{\boldsymbol{a}}_2, \hat{\boldsymbol{a}}_1\big\}$$

and introducing

$$\hat{\boldsymbol{\xi}}_{\perp\perp} \equiv (\hat{\boldsymbol{\xi}} \times \hat{\boldsymbol{\xi}}_\perp)^*$$

it is elementary that

$$(\hat{\boldsymbol{\xi}} \times \hat{\boldsymbol{\xi}}_\perp)\cdot\hat{\boldsymbol{\xi}} = (\hat{\boldsymbol{\xi}} \times \hat{\boldsymbol{\xi}}_\perp)\cdot\hat{\boldsymbol{\xi}}_\perp = 0$$

and

$$\begin{aligned} \hat{\boldsymbol{\xi}}^*_{\perp\perp}\cdot\hat{\boldsymbol{\xi}}_{\perp\perp} = (\hat{\boldsymbol{\xi}} \times \hat{\boldsymbol{\xi}}_\perp)\cdot(\hat{\boldsymbol{\xi}} \times \hat{\boldsymbol{\xi}}_\perp)^* &= (\bar{\boldsymbol{\xi}}\cdot\boldsymbol{\xi})(\bar{\boldsymbol{\xi}}_\perp\cdot\boldsymbol{\xi}_\perp) - (\bar{\boldsymbol{\xi}}\cdot\boldsymbol{\xi}_\perp)(\bar{\boldsymbol{\xi}}_\perp\cdot\boldsymbol{\xi}) \\ &= 1\cdot 1 - 0\cdot 0 \\ &= 1 \end{aligned}$$

Thus are we (by quick calculation) led to

$$\hat{\boldsymbol{\xi}}_{\perp\perp} = i \begin{pmatrix} +p_2 \\ -p_1 \\ 0 \end{pmatrix}$$

in which connection we notice that

$$\left\{\hat{\boldsymbol{a}}, -\hat{\boldsymbol{a}}_1\right\} = \begin{pmatrix} e^{+i\psi}\cos\phi \begin{pmatrix} +ip_2 \\ -ip_1 \end{pmatrix} \\ e^{-i\psi}\sin\phi \end{pmatrix} \longrightarrow e^{i\psi}\cdot\hat{\boldsymbol{\xi}}_{\perp\perp} \quad \text{at } \phi = 0$$

The equation $2\phi = 0$ serves (see again Figure 1) to describe the north *pole*

$$\hat{\boldsymbol{n}} \equiv \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

of our spherical coordinate system, where the longitudinal coordinate 2ψ —which shows up above as a simple phase factor—is indeterminate. We are led thus to write $\hat{\boldsymbol{\xi}}_{\perp\perp} = \left\{\hat{\boldsymbol{n}}, -\hat{\boldsymbol{a}}_1\right\}$, and to the conclusion that

$$\left\{\hat{\boldsymbol{a}}_2, \hat{\boldsymbol{a}}_1\right\}, \quad \left\{-\hat{\boldsymbol{a}}_2, \hat{\boldsymbol{a}}_1\right\}, \quad \left\{\hat{\boldsymbol{n}}, -\hat{\boldsymbol{a}}_1\right\}$$

comprise—for any $\hat{\boldsymbol{a}}_1$ and any $\hat{\boldsymbol{a}}_2$—a "companionable" orthonormal system in $\mathcal{C}_3$. The pattern of events is clearer in $\mathcal{C}_4$, where the orthonormality of

$$\left\{\hat{\boldsymbol{a}}_3, \hat{\boldsymbol{a}}_2, \hat{\boldsymbol{a}}_1\right\}, \quad \left\{-\hat{\boldsymbol{a}}_3, \hat{\boldsymbol{a}}_2, \hat{\boldsymbol{a}}_1\right\}, \quad \left\{\hat{\boldsymbol{n}}, -\hat{\boldsymbol{a}}_2, \hat{\boldsymbol{a}}_1\right\}, \quad \left\{\hat{\boldsymbol{n}}, \hat{\boldsymbol{n}}, -\hat{\boldsymbol{a}}_1\right\}$$

is readily verified.[2] Note the sense in which

the 2-dimensional pattern nests within
the 3-dimensional pattern, which nests within
the 4-dimensional pattern, etc.

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[2] A quartet of exceptional simplicity results (not surprisingly) from setting $\hat{\boldsymbol{a}}_1 = \hat{\boldsymbol{a}}_2 = \hat{\boldsymbol{a}}_3 = \hat{\boldsymbol{n}}$; we find

$$\left\{\hat{\boldsymbol{n}}, \hat{\boldsymbol{n}}, \hat{\boldsymbol{n}}\right\} = (\text{phase factor})\cdot\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{with} \quad (\text{phase factor}) = e^{i(\psi_3+\psi_2+\psi_1)}$$

$$\left\{-\hat{\boldsymbol{n}}, \hat{\boldsymbol{n}}, \hat{\boldsymbol{n}}\right\} = (\text{phase factor})\cdot\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{with} \quad (\text{phase factor}) = -ie^{-i\psi_3}$$

$$\left\{\hat{\boldsymbol{n}}, -\hat{\boldsymbol{n}}, \hat{\boldsymbol{n}}\right\} = (\text{phase factor})\cdot\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{with} \quad (\text{phase factor}) = -ie^{i(\psi_3-\psi_2)}$$

$$\left\{\hat{\boldsymbol{n}}, \hat{\boldsymbol{n}}, -\hat{\boldsymbol{n}}\right\} = (\text{phase factor})\cdot\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{with} \quad (\text{phase factor}) = -ie^{i(\psi_3+\psi_2-\psi_1)}$$

I use the term "companionable" to suggest "naturally associated with, but not unique." Non-uniqueness arises as follows (I work in language specific to the 4-dimensional case): Let $\mathbb{U}$ be any 4×4 unitary matrix with the property that

$$\mathbb{U}\mathbb{P} = \mathbb{P}\mathbb{U}, \quad \text{where } \mathbb{P} \text{ projects onto } \big\{\,\hat{\boldsymbol{a}}_3, \hat{\boldsymbol{a}}_2, \hat{\boldsymbol{a}}_1\big\}$$

Then the vectors

$$\Big\{\,\mathbb{U}\big\{-\hat{\boldsymbol{a}}_3, \hat{\boldsymbol{a}}_2, \hat{\boldsymbol{a}}_1\big\},\quad \mathbb{U}\big\{\,\hat{\boldsymbol{n}}, -\hat{\boldsymbol{a}}_2, \hat{\boldsymbol{a}}_1\big\},\quad \mathbb{U}\big\{\,\hat{\boldsymbol{n}}, \hat{\boldsymbol{n}}, -\hat{\boldsymbol{a}}_1\big\}\Big\}$$

are *in all cases* orthonormal in the $\mathcal{C}_3 \perp \big\{\,\hat{\boldsymbol{a}}_3, \hat{\boldsymbol{a}}_2, \hat{\boldsymbol{a}}_1\big\}$.

It is instructive to *start* with $\hat{\xi} = \mathbb{U}\big\{\,\hat{\boldsymbol{n}}, \hat{\boldsymbol{n}}, -\hat{\boldsymbol{a}}_1\big\}$ and pursue *in reverse* the procedure described above; one obtains

$$\begin{aligned}
\big\{\,\hat{\boldsymbol{n}}, \hat{\boldsymbol{n}}, -\hat{\boldsymbol{a}}_1\big\}& \\
\downarrow\quad & \\
\big\{\,\hat{\boldsymbol{n}}, -\hat{\boldsymbol{a}}_2, \hat{\boldsymbol{a}}_1\big\} &\quad : \quad \hat{\boldsymbol{a}}_2 \text{ arbitrary (2 degrees of freedom)}\\
\downarrow\quad & \\
\big\{-\hat{\boldsymbol{a}}_3, \hat{\boldsymbol{a}}_2, \hat{\boldsymbol{a}}_1\big\} &\quad : \quad \hat{\boldsymbol{a}}_3 \text{ arbitrary (2 more degrees of freedom)}\\
\downarrow\quad & \\
\big\{\,\hat{\boldsymbol{a}}_3, \hat{\boldsymbol{a}}_2, \hat{\boldsymbol{a}}_1\big\}&
\end{aligned}$$

Writing $\hat{\boldsymbol{a}}_2 = \mathbb{R}_2\hat{\boldsymbol{n}}$ and $\hat{\boldsymbol{a}}_3 = \mathbb{R}_3\hat{\boldsymbol{n}}$, there would appear to be two independent copies of $O(3)$ built into such a procedure. The theory begins at this point to acquire a distinctly "epicyclic" odor: Ptolemy meets Cartan. To say the same thing another way, and but vividly: the transformation that sends

$$\begin{aligned}
\big\{\,\hat{\boldsymbol{a}}_3, \hat{\boldsymbol{a}}_2, \hat{\boldsymbol{a}}_1\big\} &\longrightarrow \big\{\,\mathbb{R}_3\hat{\boldsymbol{a}}_3, \mathbb{R}_2\hat{\boldsymbol{a}}_2, \mathbb{R}_1\hat{\boldsymbol{a}}_1\big\}\\
\big\{-\hat{\boldsymbol{a}}_3, \hat{\boldsymbol{a}}_2, \hat{\boldsymbol{a}}_1\big\} &\longrightarrow \big\{-\mathbb{R}_3\hat{\boldsymbol{a}}_3, \mathbb{R}_2\hat{\boldsymbol{a}}_2, \mathbb{R}_1\hat{\boldsymbol{a}}_1\big\}\\
\big\{\,\hat{\boldsymbol{n}}, -\hat{\boldsymbol{a}}_2, \hat{\boldsymbol{a}}_1\big\} &\longrightarrow \big\{\,\mathbb{R}_3\hat{\boldsymbol{n}}, -\mathbb{R}_2\hat{\boldsymbol{a}}_2, \mathbb{R}_1\hat{\boldsymbol{a}}_1\big\}\\
\big\{\,\hat{\boldsymbol{n}}, \hat{\boldsymbol{n}}, -\hat{\boldsymbol{a}}_1\big\} &\longrightarrow \big\{\,\mathbb{R}_3\hat{\boldsymbol{n}}, \mathbb{R}_2\hat{\boldsymbol{n}}, -\mathbb{R}_1\hat{\boldsymbol{a}}_1\big\}
\end{aligned}$$

is necessarily unitary. In the 2-dimensional case

$$\begin{aligned}
\big\{\,\hat{\boldsymbol{a}}_1\big\} &\longrightarrow \big\{\,\mathbb{R}_1\hat{\boldsymbol{a}}_1\big\} = \mathbb{U}\big\{\,\hat{\boldsymbol{a}}_1\big\}\\
\big\{-\hat{\boldsymbol{a}}_1\big\} &\longrightarrow \big\{-\mathbb{R}_1\hat{\boldsymbol{a}}_1\big\} \quad : \quad \text{redundant}
\end{aligned}$$

we recover the $SU(2)$ representation of $O(3)$, but in all higher-dimensional cases it is clear already on numerical grounds that something funny is going on.

3. Contact with some standard representation theory. From[3]

$$\begin{aligned}
a_1 &= \quad\;\; u_1\bar{u}_1 - u_2\bar{u}_2 \;\; = \xi^{\mathsf{t}}\,\mathbb{S}_1\,\xi\\
a_2 &= \quad\;\; u_1\bar{u}_2 + u_2\bar{u}_1 \;\; = \xi^{\mathsf{t}}\,\mathbb{S}_2\,\xi\\
a_3 &= -i(u_1\bar{u}_2 - u_2\bar{u}_1) = \xi^{\mathsf{t}}\,\mathbb{S}_3\,\xi
\end{aligned}$$

---

[3] When last on stage (near the end of §1) the actors in the following equations wore hats; because $u_1\bar{u}_1 + u_2\bar{u}_2 = 1$ is now *not* presumed, those have been removed.

in follows that

$$
\begin{aligned}
a_1^2 + a_2^2 + a_3^2 &= (u_1\bar u_1 - u_2\bar u_2)^2 + (u_1\bar u_2 + u_2\bar u_1)^2 - (u_1\bar u_2 - u_2\bar u_1)^2\\
&= (u_1\bar u_1 + u_2\bar u_2)^2\\
&= \xi^{\mathrm{t}}\xi \quad\text{with}\quad \xi \equiv \begin{pmatrix} u_1\\ u_2\end{pmatrix}
\end{aligned}
$$

Write

$$
\mathbb{S} \equiv \begin{pmatrix} \alpha & \beta\\ -\beta^* & \alpha^*\end{pmatrix} \quad\text{with}\quad \alpha^*\alpha + \beta^*\beta = 1
$$

to describe a typical element of $SU(2)$. And write

$$
\xi(\tfrac{1}{2}) \equiv \begin{pmatrix} u_1\\ u_2\end{pmatrix},\ \xi(1) \equiv \begin{pmatrix} u_1^2\\ u_1u_2\\ u_2^2\end{pmatrix},\ \xi(\tfrac{3}{2}) \equiv \begin{pmatrix} u_1^3\\ u_1^2u_2\\ u_1u_2^2\\ u_2^3\end{pmatrix},\ \xi(2) \equiv \begin{pmatrix} u_1^4\\ u_1^3u_2\\ u_1^2u_2^2\\ u_1u_2^3\\ u_2^4\end{pmatrix},\ \ldots
$$

according to which scheme the $2\ell+1$ elements of $\xi(\ell)$ are stacked binomials of degree 2ℓ: $\ell = \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \ldots$ In the standard representation theory of $O(3)$[4] one is motivated to study the linear transformations

$$
\xi(\ell) \to \xi'(\ell) \quad\text{induced by}\quad \xi(\tfrac{1}{2}) \to \xi'(\tfrac{1}{2}) = \mathbb{S}\,\xi(\tfrac{1}{2})
$$

One finds, for example, that

$$
\xi'(1) = \underbrace{\begin{pmatrix} \alpha^2 & 2\alpha\beta & \beta^2\\ -\alpha\beta^* & (\alpha^*\alpha - \beta^*\beta) & \beta\alpha^*\\ \beta^{*2} & -\alpha^*\beta^* & \alpha^{*2}\end{pmatrix}}_{\mathbb{S}(1)}\xi(1)
$$

The matrices $\mathbb{S}(1)$ and $\mathbb{R}$ share more than the facts that both are 3×3, both derived from the structure of $\mathbb{S}(\frac{1}{2}) \equiv \mathbb{S}$, both become $\mathbb{I}_3$ in the limit $\mathbb{S} \to \mathbb{I}_2$; from the line of argument developed in the material to which I just referred it becomes fairly natural to notice that the complex 3-vector defined

$$
\boldsymbol{o} \equiv \tfrac{1}{2}\begin{pmatrix} (u_1^2 + u_2^2)\\ -i(u_1^2 - u_2^2)\\ 2iu_1u_2\end{pmatrix} = \mathbb{C}\xi(1) \quad\text{with}\quad \mathbb{C} \equiv \tfrac{1}{2}\begin{pmatrix} 1 & 0 & 1\\ -i & 0 & +i\\ 0 & 2i & 0\end{pmatrix}
$$

is a *null* vector:

$$
\boldsymbol{o}\cdot\boldsymbol{o} = 0 + i0
$$

And that $\boldsymbol{o}$ transforms

$$
\boldsymbol{o} \longrightarrow \boldsymbol{o}' = \mathbb{R}\boldsymbol{o}
$$

where

$$
\mathbb{R} \equiv \mathbb{C}\,\mathbb{S}(1)\mathbb{C}^{-1}
$$

[4] See §2 of "Applications of the theory of harmonic polynomials" (1996).

where $\mathbb{R}$ is precisely the rotation matrix associated with $\mathbb{S}$, and to which the "Cayler-Klein parameters" $\{\alpha, \beta\}$ refer. The matrix $\mathbb{S}(1)$ turns out to be "unitary" only in this generalized (and dimensionally generalizable) sense:

$$\mathbb{S}^{\mathsf{t}}\mathbb{G}\mathbb{S} = \mathbb{G} \quad \text{where} \quad \mathbb{G} \equiv \mathbb{C}^{\mathsf{t}}\mathbb{C} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$$
$$= \text{“induced metric” in the spin-space where } \xi(1) \text{ lives}$$

Much of thet mystery is removed from preceding assertions by the elementary observation that the $\mathbb{S}(1)$-invariant expression

$$\xi(1)^{\mathsf{t}}\mathbb{G}\,\xi(1) \quad \text{can be written} \quad \tfrac{1}{2}(\bar{u}_1 u_1 + \bar{u}_2 u_2)^2$$

We note in passing that the theory sketched above does not concern itself with the construction of complete orthonormal sets (whatever the generalized meaning we might want to assign to that notion), and that it assigns no explicit importance to projection matrices of any description.

Contrast the preceding theory with that which unfolds when similar ideas are brought to bear on the hyperspherical "Chinese box formalism" discussed previously. Confining my explicit remarks to the 3-dimensional case, let us in the first instance suppose that $\mathbb{R}$ is active only within the innermost box:

$$\{\hat{\boldsymbol{a}}_2, \hat{\boldsymbol{a}}_1\} \longrightarrow \{\hat{\boldsymbol{a}}_2, \mathbb{R}\hat{\boldsymbol{a}}_1\}$$

This we express

$$\begin{pmatrix} q_1 p_1 \\ q_1 p_2 \\ q_2 \end{pmatrix} \longrightarrow \begin{pmatrix} q_1(\alpha p_1 + \beta p_2) \\ q_1(-\beta^* p_1 + \alpha^* p_2) \\ q_2 \end{pmatrix} = \underbrace{\begin{pmatrix} \alpha & \beta & 0 \\ -\beta^* & \alpha^* & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\mathbb{U}} \begin{pmatrix} q_1 p_1 \\ q_1 p_2 \\ q_2 \end{pmatrix}$$

which is not very interesting: $\mathbb{U}$ simply reproduces within $SU(3)$ a copy of $SU(2)$. But suppose $\mathbb{R}$ is active (only) within the *next*-inner box:

$$\{\hat{\boldsymbol{a}}_2, \hat{\boldsymbol{a}}_1\} \longrightarrow \{\mathbb{R}\hat{\boldsymbol{a}}_2, \hat{\boldsymbol{a}}_1\}$$

We then have

$$\begin{pmatrix} q_1 p_1 \\ q_1 p_2 \\ q_2 \end{pmatrix} \longrightarrow \begin{pmatrix} (\alpha q_1 + \beta q_2) p_1 \\ (\alpha q_1 + \beta q_2) p_2 \\ (-\beta^* q_1 + \alpha^* q_2) \end{pmatrix}$$

which (since terms appear on the right which are not present on the left) *can be accomplished by no* 3×3 *matrix!* Nor (when one looks to the hyperspherical description of the elements of the vector) is the loss of linear closure particularly surprising. The obvious way to recover linear closure is to "augment the stack," looking to

$$\begin{pmatrix} q_1p_1 \\ q_2p_1 \\ q_1p_2 \\ q_2p_2 \\ q_1 \\ q_2 \end{pmatrix} \longrightarrow \begin{pmatrix} \alpha & \beta & & & & \\ -\beta^* & \alpha^* & & & & \\ & & \alpha & \beta & & \\ & & -\beta^* & \alpha^* & & \\ & & & & \alpha & \beta \\ & & & & -\beta^* & \alpha^* \end{pmatrix} \begin{pmatrix} q_1p_1 \\ q_2p_1 \\ q_1p_2 \\ q_2p_2 \\ q_1 \\ q_2 \end{pmatrix}$$

The "innermost" transformation $\{\hat{\boldsymbol{a}}_2, \hat{\boldsymbol{a}}_1\} \longrightarrow \{\hat{\boldsymbol{a}}_2, \mathbb{R}\hat{\boldsymbol{a}}_1\}$ acquires, in this expanded setting, the representation

$$\begin{pmatrix} q_1p_1 \\ q_2p_1 \\ q_1p_2 \\ q_2p_2 \\ q_1 \\ q_2 \end{pmatrix} \longrightarrow \begin{pmatrix} \alpha & 0 & \beta & 0 & & \\ 0 & \alpha & 0 & \beta & & \\ -\beta^* & 0 & \alpha^* & 0 & & \\ 0 & -\beta^* & 0 & \alpha^* & & \\ & & & & 1 & 0 \\ & & & & 0 & 1 \end{pmatrix} \begin{pmatrix} q_1p_1 \\ q_2p_1 \\ q_1p_2 \\ q_2p_2 \\ q_1 \\ q_2 \end{pmatrix}$$

and when the preceding statements are conflated we find ourselves discussing the representation within $SU(6)$ of $O(3) \times O(3)$. Here the 6 has obvious origin in the circumstance that $2+2^2 = 6$. Study of $\{\hat{\boldsymbol{a}}_3, \hat{\boldsymbol{a}}_2, \hat{\boldsymbol{a}}_1\}$ by such means would lead to $SU(N)$ with $N = 2+2(6) = 2+2^2+2^3 = 14$. The numerology proceeds

$$\begin{aligned} 2 = 2^2 - 2 \quad &: \quad \text{case } \xi \in \mathcal{C}_2 \\ 2+2(2) = 2+2^2 = \;\; 6 = 2^3 - 2 \quad &: \quad \text{case } \xi \in \mathcal{C}_3 \\ 2+2(6) = 2+2^2+2^3 = 14 = 2^4 - 2 \quad &: \quad \text{case } \xi \in \mathcal{C}_4 \\ 2+2(14) = 2+2^2+2^3+2^4 = 30 = 2^5 - 2 \quad &: \quad \text{case } \xi \in \mathcal{C}_5 \\ &\;\vdots \end{aligned}$$